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COMMENT

A spinning particle in a magnetic field and the superharmonic oscillator

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Abstract. The system of a spinning particle in a general magnetic field will be shown to be supersymmetric, having two supersymmetry generators. We will discuss the quantisation of this system in the case of a constant magnetic field, and we will describe the states by means of wavefunctionals, having a superfield as an argument.

One of the difficulties related to the superspace description of a supersymmetric theory is how to write a wavefunctional that depends on a superfield. The reasons for this difficulty are as follows. 1. In the fermionic sector the superfield uses the whole fermionic phase space, so the set of superfields on a space-like surface is not the correct polarised set, or the superfield in two superspace points has a non-vanishing equal time commutator [1]. 2. For a linear representation of supersymmetry, one uses auxiliary fields that obey constraint equations, so the superfield does not behave as an unconstrained coordinate in the canonical formalism. One possible solution for this problem could be a description of the state by means of Wigner distributions that are defined over the whole phase space, but this does not solve the possibility to develop canonical formalism and wavefunctionals. A simple system that we will study is the superharmonic oscillator (SHO) [2]. Physically, the SHO is related to the spinning particle in a constant magnetic field [3, 4], and before the SHO we will study a more general case. A spinning, charged, non-relativistic free particle can be described by the following Lagrangian:

$$L = \frac{i}{2}\,\xi_i\dot{\xi}_i + \tfrac{1}{2}\dot{x}_i^2 \tag{1}$$

where x_i are the translational coordinates, and ξ_i are the spin variables (classically $\{\xi_i, \xi_j\} = 0$ and quantum mechanically $\{\xi_i, \xi_j\} = \delta_{ij}$). The Hamiltonian of this system is:

$$H = \tfrac{1}{2}P_i^2 = (\Sigma \pm P_i\xi_j)^2.$$

The system enjoys supersymmetry that is generated by 18 generators, namely (six different $P_i\xi_j$) $\times$ (three independent possibilities of relative signs). When we add a magnetic field, the Hamiltonian is

$$H = \tfrac{1}{2}(\boldsymbol{P} - \boldsymbol{A})^2 + \frac{i}{2}\,\varepsilon_{ijk}B_k\xi_i\xi_j \tag{2}$$

where $\boldsymbol{B} = \text{curl}\,\boldsymbol{A}$. Define

$$A_{\|} = \text{the projection of } \boldsymbol{A} \text{ along } \boldsymbol{B}$$

$$A_{\perp} = \boldsymbol{A} - A_{\|}$$

so $\boldsymbol{A} \times \boldsymbol{B}$, $\boldsymbol{B}$ and $A_{\perp}$ form a Cartesian frame. By referring to this frame, one can write the Hamiltonian in the following way:

$$H = \tfrac{1}{2}[(P_{\perp} - A_{\perp})^2 + (P_{\boldsymbol{A}\times\boldsymbol{B}})^2 + (P_{\|} - A_{\|})^2] + \mathrm{i}|B|\xi_{\perp}\xi_{\boldsymbol{A}\times\boldsymbol{B}} \tag{3}$$

or in these inequivalent ways:

$$H = [(P_{\perp} - A_{\perp})\xi_{\perp} + (P_{\boldsymbol{A}\times\boldsymbol{B}})\xi_{\boldsymbol{A}\times\boldsymbol{B}} \pm (P_{\|} - A_{\|})\xi_{\|}]^2 \equiv [Q_1 \pm \xi_{\|}(P_{\|} - A_{\|})]^2 \tag{4}$$

and

$$H = [(P_{\perp} - A_{\perp})\xi_{\boldsymbol{A}\times\boldsymbol{B}} - (P_{\boldsymbol{A}\times\boldsymbol{B}})\xi_{\perp} \pm (P_{\|} - A_{\|})\xi_{\|}]^2 \equiv [Q_2 \pm \xi_{\|}(P_{\|} - A_{\|})]^2. \tag{5}$$

This means that the system has four generators of supersymmetry†. From now on we will discuss the case of a constant, z-directed magnetic field, and one can easily see that the motion in the $\|$ direction can be decoupled, so $H \to H - \frac{1}{2}(P_{\|} - A_{\|})^2$. The system is equivalent to the SHO with the following supersymmetry algebra:

$$\{Q_i, Q_j\} = 2H\delta_{ij} \qquad [H, Q_i] = 0. \tag{6}$$

This algebra has the standard realisation [2-4] on superspace that contains: t, θ_1, θ_2. Our next step will be to express the bosonic degrees of freedom together with the fermionic ones by a common 'superfield', and to find such a field that will be a superspace dependent and a wavefunctional of this field can express the state. One can observe that the directions in superspace θ_1 and θ_2 cannot serve as a 'quantisation surface' or a space-like surface used for the usual field theoretic canonical quantisation, because two translations on this surface can give a time translation. It will be convenient to use complex generators: $Q = (1/\sqrt{2})(Q_1 + \mathrm{i}Q_2)$ and $\bar{Q} = (1/\sqrt{2})(Q_1 - \mathrm{i}Q_2)$ that obey:

$$\{Q, \bar{Q}\} = 2H \qquad Q^2 = \bar{Q}^2 = 0 = [H, \bar{Q}] = [H, Q]. \tag{7}$$

Define: $Q = a^+\psi$ and $\bar{Q} = a\psi^+$. By these variables the classical SHO has the following Lagrangian:

$$L = \frac{\mathrm{i}}{2}(\dot{\bar{a}}a - \dot{a}\bar{a}) - \frac{m}{2}\{a, \bar{a}\} + \frac{\mathrm{i}}{2}(\dot{\bar{\psi}}\psi + \dot{\psi}\bar{\psi}) - \frac{m}{2}[\psi, \bar{\psi}]. \tag{8}$$

On the qantum level:

$$[a^+, a] = \{\psi^+, \psi\} = 1$$

all the other relations are zero. The action is invariant under supersymmetry generated by $Q = a\psi^+$ and $Q^+ = a^+\psi$. Namely

$$\delta\psi = \varepsilon a \qquad \delta^+\psi^+ = \varepsilon^+ a^+ \qquad \delta^+ a = -\varepsilon^+\psi \qquad \delta a^+ = \varepsilon\psi^+.$$

According to (7), one can choose a 'space-like' line along the θ ($= \theta_1 + \mathrm{i}\theta_2$) direction and quantise along this line.

† This result is slightly different from that of [3] where the magnetic field has a constant direction and the vector potential is in the form $A_x = -2yW'(x^2+y^2)$, $A_y = 2xW'(x^2+y^2)$.

Define: $W(t, \theta) = a + \theta\psi$ and its (*) conjugate: $W^* = \theta a^* - \psi$. W is a bosonic variable, and W^* is fermionic. The (*) action is defined [5]: $f^*(\theta) = (a + b\theta)^* = \int d\theta_1 \bar{f}(\theta_1)\, e^{\theta_1\theta}$ where ($\bar{\cdot}$) is the usual complex conjugation. The action in terms of W and W^* is

$$S = \int d\theta\, dt \left(\frac{i}{2} (\dot{W}^* W - \dot{W} W^*) - \frac{m}{2} \{W^*, W\} \right). \tag{9}$$

The canonical momenta are

$$\frac{\delta S}{\delta \dot{W}} = -\frac{i}{2} W^* \qquad \frac{\delta S}{\delta \dot{W}^*} = \frac{i}{2} W.$$

This leads to the canonical commutation relations for the quantum operators corresponding to W and W^*:

$$[W(\theta), W^+(\theta')] = \delta(\theta - \theta') \tag{10}$$

where $^+$ stands for Hermitian conjugation. The realisation of this relation will be [6]: $W^+ \to \delta/\delta Z(\theta)$ and $W \to Z(\theta)$, and Z is defined on the superspace t, θ as $Z = X + \theta\eta$.

The Hamiltonian is

$$\begin{gathered} H = m \int d\theta\, W W^+ = m \int d\theta\, Z \frac{\delta}{\delta Z} \\ H\Psi(Z) = E\Psi(Z) \Rightarrow \Psi_n^a = \int d\theta Z^n \qquad \Psi_n^b = \int d\theta\theta Z^n \end{gathered} \tag{11}$$

and the both eigenfunctions have the energy nm. One can easily verify that the supersymmetry generators Q and $\bar{Q}$ relate between Ψ_a^n and Ψ_b^n $(n > 0)$. The ground state is $\Psi_0 = 1$, and is non-degenerate.

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